Three Reasons to Price Carbon under Uncertainty: Accuracy of Simple Rules

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Climate change and financial markets

- Climate change is caused by greenhouse gas emissions.
- Higher temperatures negatively affect economic activity (Dell et al. 2009, 2012; Burke at al. 2015, Jägermeyer et al. 2021).
  - Loss of aggregate output
  - Increased risk of climate-related disasters (e.g., hurricanes, floods, droughts)
  - Hazard of climate tipping
- There is recent evidence that climate change also affects **financial markets**.
  - Temperature fluctuations carry a risk premium in equity markets (Donadelli et al. 2017; Bansal et al. 2019).
  - Carbon-intensive firms carry a risk premium (Donadelli et al. 2019; Bolton and Kacperczyk 2021).
- To mitigate those impacts, emissions-free technologies and renewable energies are developed and carbon dioxide emissions are priced.

Research question

- The social cost of carbon (SCC) is the (expected) present value of all future damages stemming from emitting one ton of CO_2 into the atmosphere and the optimal Pigouvian tax that internalizes the negative externalities from emitting CO_2.

Research Question

1. How do climate-related externalities affect the SCC?
2. How does climate change affect asset prices?
3. Can we find a tractable, closed-form rule for the SCC?
4. How accurate is the closed-form rule?

- We address these research questions in a simple DSGE model.
- We exploit perturbation theory to derive interpretable formulas for the SCC and asset pricing moments.
- We show that this approximation works very well even if the economy is driven by various risk sources.

Integrated assessment models

- An IAM is a model that combines dynamics of production, carbon dioxide emissions, and climate change in a unified framework to determine the social cost of carbon.
- The workhorse example for an optimization-based IAM is the DICE model (Nordhaus 1992, 2017).
- Important Issue: How to calibrate the preference parameters of these models?
  - Stern argues that the discount rate should be very small because of ethical reasons.
  - Nordhaus argues that the discount rate should be higher since it should be in line with what we do observe in reality.
- In deterministic IAMs, we cannot draw any conclusions about financial markets.

Closed-form solutions

- There is a huge literature that provides closed-form solutions for the social cost of carbon in *simplified settings*:
    - Nordhaus (1991), EJ
    - Golosov et al. (2014), Econometrica
    - Rezai and van der Ploeg (2016), JAERE
    - van den Bijgaart et al. (2016), JEEM
    - Traeger (2020), AEJ Macro
    - Hillebrand and Hillebrand (2021), JET
    - Hambel et al. (2021), JIE
- Deriving closed-form solutions requires debatable assumptions on the the preferences (e.g., log-utility) or the production system (e.g., full depreciation).
- Another idea to come up with closed-form solutions is to keep the model complicated but to *expand* the solution in an analytical expression, e.g.,
    - van den Bremer and van der Ploeg (2021), AER

Asset pricing models

- Risk-free Rate puzzle / Equity Premium Puzzle:
  It is hard to provide a model-based explanation for the low risk-free rate ($\mathbb{E}[r_f] \approx 0.8\%$) and the high equity premium ($ep = \mathbb{E}[r_m - r_f] \approx 6.5\%$).
- Some solution ideas
  - Habit formation: Campbell and Cochrane (1999), JPE; Abel (1999), JME
  - Long-run risk and recursive utility: Bansal and Yaron (2004), JF
  - Disaster risk (and recursive utility): Barro (2006), QJE; Barro (2009), AER; Wachter (2013), JF, Pindyck and Wang (2013), AEJ EP
- Idea: Equip an integrated assessment model with economic (and climate) uncertainty and calibrate the preference parameters such that the model matches the empirically observed asset pricing moments.

Stochastic IAM

- Capital stock (with regular GBM uncertainty and volatility σ and disaster shocks with intensities λ_e and $\lambda_c(T)$) and losses ℓ_e and ℓ_c:

$$\mathrm{d}K = \left[I - \delta K - \frac{1}{2}\varphi\frac{I^2}{K}\right]\mathrm{d}t + \sigma K \mathrm{d}W_e - K_{-}\ell_e\mathrm{d}N_e - K_{-}\ell_c\mathrm{d}N_c$$

- Output (AK production function):

$$Y = AK^{\alpha}F^{1-\alpha}D(T) = I + C + bF$$

- Temperature depends on cumulative emissions of carbon dioxide:

$$\mathrm{d}T = \chi\mathrm{d}E + \sigma_T\mathrm{d}W_T, \qquad \mathrm{d}E = \xi F\mathrm{d}t$$

- Duffie-Epstein preferences:

$$\mathcal{U}_t = \mathbb{E}_t\left[\int_t^{\infty} f(C_s, \mathcal{U}_s)\mathrm{d}s\right], \quad f(C, J) = \rho\theta J\left[\left(\frac{C^{1-1/\psi}}{[(1-\gamma)J]^{1/\theta}}\right) - 1\right]$$

Stochastic optimization problem

- The social planner chooses optimal consumption and fossil fuel use to maximize expected utility, i.e.,

$$J_t = \sup_{C,F} \mathcal{U}_t$$

- The indirect utility function J satisfies a Hamilton-Jacobi-Bellman equation, which does not have a closed-form solution

$$\max_{C,F} \Big\{ f(C,J) + J_K \Big[AK^{\alpha} F^{1-\alpha} D(T) - C - bF - \delta K - 0.5\varphi \frac{I^2}{K} \Big] + \frac{1}{2} J_{KK} K^2 \sigma^2 + J_T \chi \xi F + \frac{1}{2} J_{TT} \sigma_T^2 \sum_{i=e,c} \lambda_i(T) \mathbb{E}\Big[J\big((1-l_i)K, T\big) - J \Big] \Big\} = 0.$$

- We compare two solution approaches: standard grid-based numerical optimization approach and analytical perturbation method

Perturbation method I

- Perturbation theory is a method for finding an approximate solution to a complicated problem by starting with the *exact* solution of a related, simpler problem
- In our case, the problem without climate externalities has a closed-form solution denoted by $J^{(0)}$ (see Pindyck and Wang 2013)
- Climate change disturbs this closed-form solution
- However, as long as the impact of climate change ε is *small*, the error between the true indirect utility function J and the zeroth-order term $J^{(0)}$ might be small as well
- Idea is to expand the indirect utility function J in a series with terms of increasing order in ε

Perturbation method II

- The full complicated problem is thus not solved exactly, but instead small correction terms are added to adjust the solution of the simpler, exactly solvable problem $J^{(0)}$:

$$J = \sum_{k=0}^{\infty} \varepsilon^k J^{(k)}$$

- We find that for our purposes it is sufficient to expand the indirect utility function to the first-order term $J^{(1)}$, which captures most of the disturbance stemming from climate change, i.e., we determine

$$J \approx J^{(0)} + \varepsilon J^{(1)}$$

- Using this approximation, we derive the optimal SCC as the marginal rate of substitution between capital and emissions
- In this model, we have $J = \psi(T)K^{1-\gamma}$; $J^{(0)} = \psi_0(T_0)K^{1-\gamma}$; $J^{(1)} = \psi_1(T_0)K^{1-\gamma}$; and $\varepsilon = F(T)$

Results for risk-free rate and equity premium

Let $Z_i = 1 - \ell_i$ *denote recovery rates, and assume that damages and the intensity of climate-related disaster shocks are linear in temperature:*

$$D(T) = D_0 + D_1 T, \qquad \lambda_c(T) = \lambda_0 + \lambda_1 T$$

The leading-order approximation to the risk-free rate is

$$r_f = \rho + \frac{g^{(0)}}{\psi} - \frac{1}{2}\gamma\Big(\frac{1}{\psi} + 1\Big)\sigma^2 - \sum_{i=e,c} \lambda_i(T)\mathbb{E}\Big[(Z_i^{-\gamma} - 1) + \frac{1/\psi - \gamma}{1-\gamma}(1 - Z_i^{1-\gamma})\Big]$$

where $\gamma = RRA$ *and* $\psi = EIS$*. The equity premium is*

$$r_p = \gamma\sigma^2 + \sum_{i=e,c} \lambda_i(T)\mathbb{E}\big[(1 - Z_i)(Z_i^{-\gamma} - 1)\big]$$

Result 1: Rule without tipping risk

The leading-order approximation to the optimal SCC is

$$P_{\mathrm{R1}} = D_1 \frac{\chi Y_t^{(0)}}{r^\star} + \lambda_1 q \frac{\mathbb{E}[1 - Z_c^{1-\gamma}]}{1-\gamma} \frac{\chi K_t}{r^\star},$$

where the discount rate adjusted for risk and growth is given by

$$\begin{aligned} r^\star &= \rho + \left(\frac{1}{\psi} - 1\right)\left(g^{(0)} - \frac{1}{2}\gamma\sigma^2 + \sum_{i=e,c} \frac{\lambda_i(T)}{1-\gamma}\mathbb{E}[1 - Z_i^{1-\gamma}]\right) \\ &= r_f + r_p - \left(g^{(0)} - \sum_{i=e,c} \lambda_i(T)\mathbb{E}[1 - Z_i]\right), \end{aligned}$$

and Tobin's Q is

$$q = \frac{1}{1 - \varphi \frac{I}{K}}.$$

Remark: Easy to add a multitude of different climate-related disasters.

Extension: Climate tipping points

- There are tipping points in the Earth's climate system, e.g., albedo feedback or permafrost melting.
- These tipping points—once activated—yield a sudden shock to the global average temperature and, in turn, to the economy.
- Carbon dioxide and temperature with climate tipping:

$$\mathrm{d}T = \chi \mathrm{d}E + \sigma_c \mathrm{d}W_c, \qquad \mathrm{d}E = \xi F \mathrm{d}t$$

  where χ follows a Markov chain.
- We consider the special case of a two-stated Markov chain for with a pre-tipping state χ_0 and a post-tipping state $\overline{\chi}$.
- Assume that the jump intensity is $H(T) = (h_0 + h_1 T)1_{\{\chi=\chi_0\}}$.
- The perturbation method yields additional terms in the SCC that take account of this tipping risk.

Result 2: Rule with tipping risk

The leading-order approximation to the SCC is

$$P_{\mathrm{R2}} = P_{\mathrm{R1}}\frac{\psi_0}{\psi} + \frac{h'(E)}{r^\star}\frac{Y^{(0)}q^{(0)}}{B^{(0)}}\frac{1}{1-\gamma}\left(\frac{\psi_0-\overline{\psi}_0}{\psi}\right)$$
$$+\frac{h(E)}{r^\star}\left(\overline{P}_{\mathrm{R1}}\frac{\overline{\psi}_0}{\psi} - P_{\mathrm{R1}}\frac{\psi_0}{\psi}\right),$$

where the post-tip SCC $\overline{P}_{\mathrm{R1}}$ *and the pre-tip SCC without tipping* P_{R1} *are*

$$P_{\mathrm{R1}} = \left[D_{1T} + \lambda^c_{1T}\frac{\mathbb{E}\left[1-Z_c^{1-\gamma}\right]}{1-\gamma}\frac{q^{(0)}}{B^{(0)}}\right]\frac{\chi_0 Y^{(0)}}{r^\star},$$
$$\overline{P}_{\mathrm{R1}} = \left[D_{1T} + \lambda^c_{1T}\frac{\mathbb{E}\left[1-Z_c^{1-\gamma}\right]}{1-\gamma}\frac{\overline{q}^{(0)}}{\overline{B}^{(0)}}\right]\frac{\overline{\chi}\,\overline{Y}^{(0)}}{\overline{r}^\star}$$

and the ψ's are the coefficients of the value function in the absence of tipping ψ_0, after tipping $\overline{\psi}_0$, and with consideration of tipping ψ.

Calibration

1. We calibrate the economic part and the preference parameters of the model without substantial carbon taxes such that it matches the historical average (e.g., Pindyck and Wang 2013)
   - consumption growth rate of $\approx 2\%$,
   - investment-output ratio of $\approx 25\%$,
   - real interest rate of $\approx 0.8\%$,
   - equity premium of $\approx 6.5\%$,
   - consumption volatility in normal times of $\approx 2\%$,
   - industrial carbon dioxide emissions ≈ 9.16GtC
   - ...
2. We calibrate the climate part of the model and the damage specifications such that it closely matches the evolution of the RCP8.5 scenario of the IPCC
   - transient response to cumulative emissions $\chi = 1.8^\circ\text{C/TtC}$
   - damage sensitivity $D_1 = 0.9\%/^\circ\text{C}$
   - disaster intensity $\lambda_1 = 9.6\%/^\circ\text{C}$

Numerical Results

- We compare the closed-form approximation and the numerical solution obtained from a grid-based approach
- The approximation works very well without tipping points, and reasonably well with tipping points

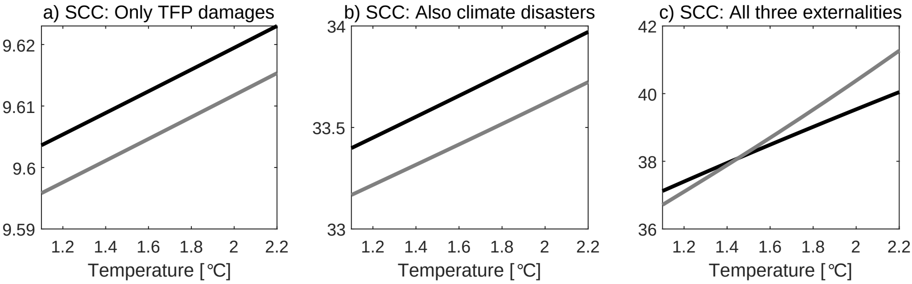
(a) Models without tipping points

Method of calculation	Rule	Grid	Error
TFP damages	9.60	9.60	-0.04%
Climate disasters	23.53	23.73	-0.85%
TFP damages and climate disasters	33.17	33.40	-0.69%

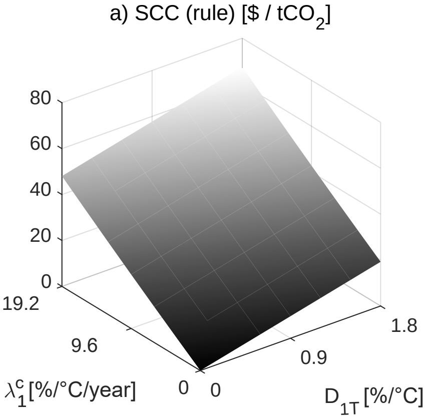
(b) Models with tipping points

Method of calculation	Rule	Grid	Error
TFP damages	10.33	10.62	-2.72%
Climate disasters	26.41	26.35	-0.23%
TFP damages and climate disasters	36.67	37.12	-1.21%

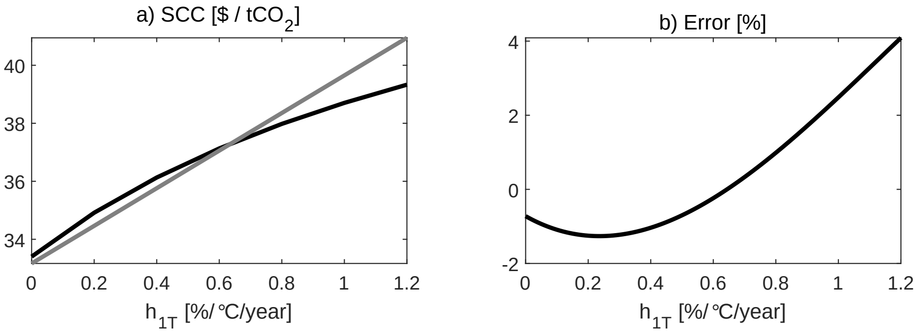
Sensitivity analysis suggests numerical errors are small



Robustness: Varying marginal arrival arrival rate of climate disasters and the marginal damage



Robustness: Varying marginal arrival rate of tipping point



Robustness: Lower discount rates

(a) Benchmark: market discount rate $r^* = 5.3\%$

Allow for:	**Rule**	**Grid**	**Error**
TFP damages	9.60	9.60	-0.04%
TFP damages and climate disasters	33.17	33.40	-0.69%
TFP damages, climate disasters, tipping	36.67	37.12	-1.21%

(b) Lower discount rate $r^* = 3\%$

Allow for:	**Rule**	**Grid**	**Error**
TFP damages	17.01	17.06	-0.28%
TFP damages and climate disasters	75.78	77.26	-1.91%
TFP damages, climate disasters, tipping	90.67	91.62	-1.04%

(c) Even lower discount rate $r^* = 2\%$

Allow for:	**Rule**	**Grid**	**Error**
TFP damages	25.47	25.63	-0.63%
TFP damages and climate disasters	139.19	143.88	-3.26%
TFP damages, climate disasters, tipping	181.87	179.50	1.32 %

Robustness: Two-sector model (Hambel et al. 2024)

(a) Benchmark: market discount rate $r^* = 5.3\%$

Allow for:	**Rule**	**Grid**	**Error**
TFP damages	9.60	9.58	0.21%
TFP damages and climate disasters	33.12	32.93	0.57%
TFP damages, climate disasters, tipping	36.67	37.28	-1.63%

(b) Lower discount rate $r^* = 3\%$

Allow for:	**Rule**	**Grid**	**Error**
TFP damages	17.06	16.90	0.94%
TFP damages and climate disasters	75.78	75.11	0.88%
TFP damages, climate disasters, tipping	90.67	88.37	2.60%

(c) Even lower discount rate $r^* = 2\%$

Allow for:	**Rule**	**Grid**	**Error**
TFP damages	25.47	25.00	1.85%
TFP damages and climate disasters	139.19	133.34	4.20%
TFP damages, climate disasters, tipping	181.87	168.75	7.77%

Extension: Uncertain and Convex Climate Damages

- Assume that the damage function $D(T)$ is not deterministic.
- Introduce a state variable μ that follows

$$\mathrm{d}\mu = \nu[\overline{\mu} - \mu]\mathrm{d}t + \sigma_\mu \mathrm{d}W_\mu$$

- The damage function is $D(T, \mu) = 1 - \max(0, \mu)^{1+\theta_\mu} T$, and the damage distribution is thus skewed.
- The pertubation method yields a correction term for the skewness of damage uncertainty.
- If the damage function $D(T, \mu)$ is convex in T, it also yields a correction term for convexity:

$$P_t = D^* \frac{\chi Y_t^{(0)}}{r^\star} + \lambda_1 q \frac{\mathbb{E}[1 - Z^{1-\gamma}]}{1-\gamma} \frac{\chi K_t}{r^\star},$$

$$D^* = \frac{\partial D}{\partial T} \left(1 + \frac{1}{2}\theta_\mu \left(1 + \theta_\mu\right) \frac{(\sigma_\mu/\bar{\mu})^2}{r^* + 2\nu} + \Upsilon \right).$$

- Comparable accuracy of the simple rules.

Conclusion

- Perturbation methods give good approximations to indirect utility functions, the SCC, and asset pricing moments in stochastic IAMs
- The solutions can be easily implemented and provide a natural decomposition into the different components affecting the SCC and asset pricing moments
- The methods lead to a natural calibration strategy that takes financial market data into account
- The error from the numerical solution is small for reasonable calibrations
- The method is robust to model extensions (e.g., convex damages, damage uncertainty, convex temperature dynamics, two sectors, non-constant energy costs, temperature/carbon stock dynamics, ...) and not plagued by the curse of dimensionality